Relativity and clocks: Connecting Theory with Practice

Image credit: SpaceX/NASA

February 12, 2025
Vienna

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Time: Newton and Einstein

~~Space and time are absolute.~~

There's only spacetime. Every point in spacetime is an event.

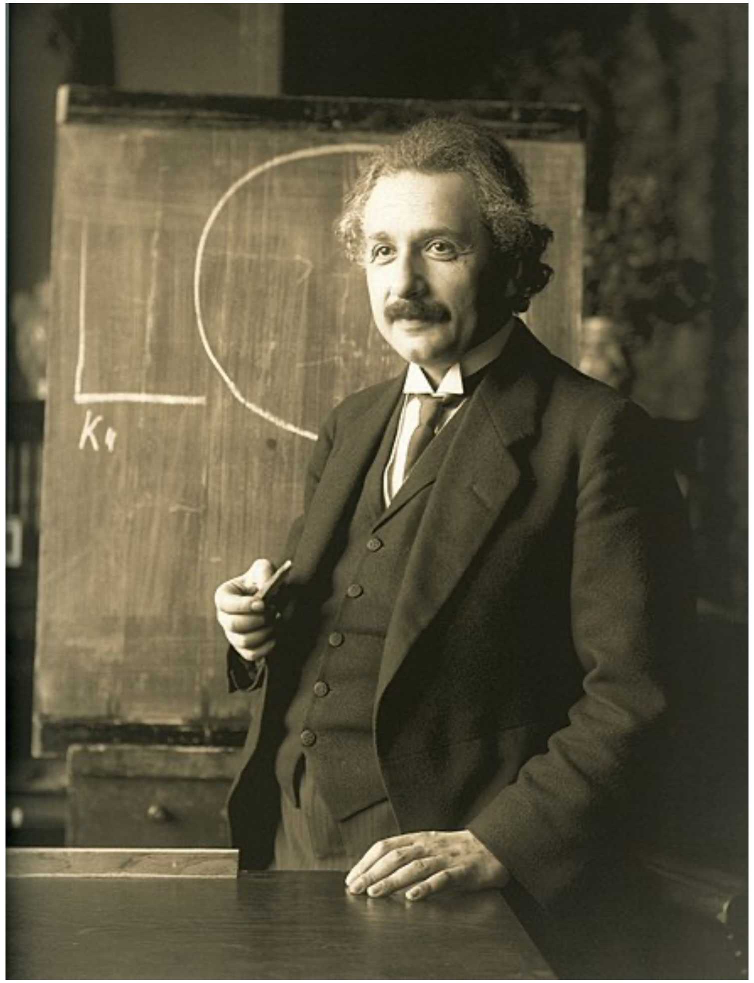
~~Space extends to infinity in all three directions, and time is the same at every point in space at any given instant~~.

Length and time intervals are relative. Clocks tell the proper time.

~~The rate of clocks is the same everywhere.~~

The motion of the clock and gravity at the location of the clocks determine the rate of clocks.

Newton's notion of space and time matches Einstein's only if the universe is empty and static.



Credit: Wikipedia

Einstein's theory of relativity

General relativity (Einstein, 1915), is based on three universality principles—collectively called the Einstein Equivalence Principle (EEP).

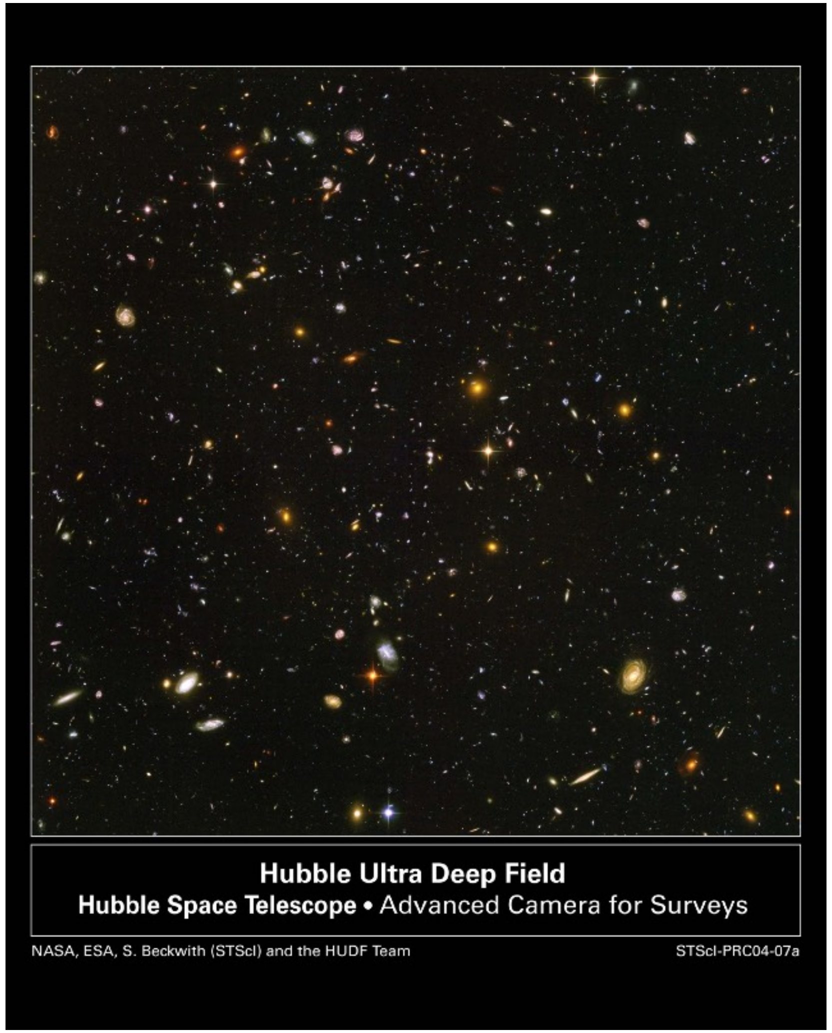
1) The universality of free fall (UFF). Also known as the Weak Equivalence Principle (WEP).
Outcomes of gravitational experiment are "same" in free fall.

2) The universality of gravitational redshift (UGR). This is a consequence of Local Position Invariance (LPI).
Outcomes of non-gravitational experiments are "same" everywhere, anytime.

3) A universality with respect to the state of observers moving with different velocities:
Local Lorentz invariance (LLI).
All observers agree with each other if their relative motions are accounted for.

Using the above postulates, how do you calculate the proper time elapsed by a moving clock?

Reference frames in Relativity: An example



You may have heard that the universe is ~13.8 billion years old.

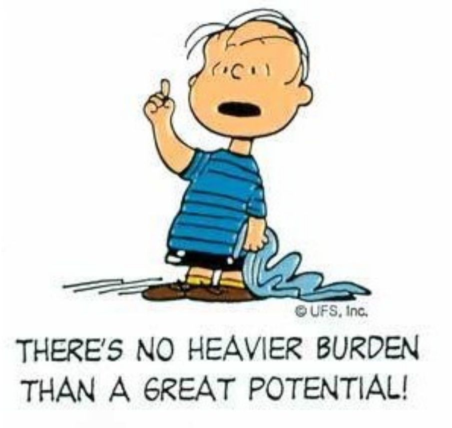
Who is the observer that determines this number?

An observer who is at rest† w.r.t. the Cosmic Microwave Background Radiation (CMBR) frame. (Solar system is moving ~370 km/s w.r.t. CMBR frame).

You need an appropriate coordinate system and a associated notion of time. But actually this is not a complicated calculation!

† Could be a fraction of a ly or few million ly depending on whether you choose to accelerate or drift.

Respecting Equivalence Principle and more…



1) Reference frames must be in free fall in the gravitational field of the central body in which test particles move.

2) External matter are to offer only tidal contributions in the second order.

3) First order tidal terms may be eliminated by translating the origin of the local coordinate frame.

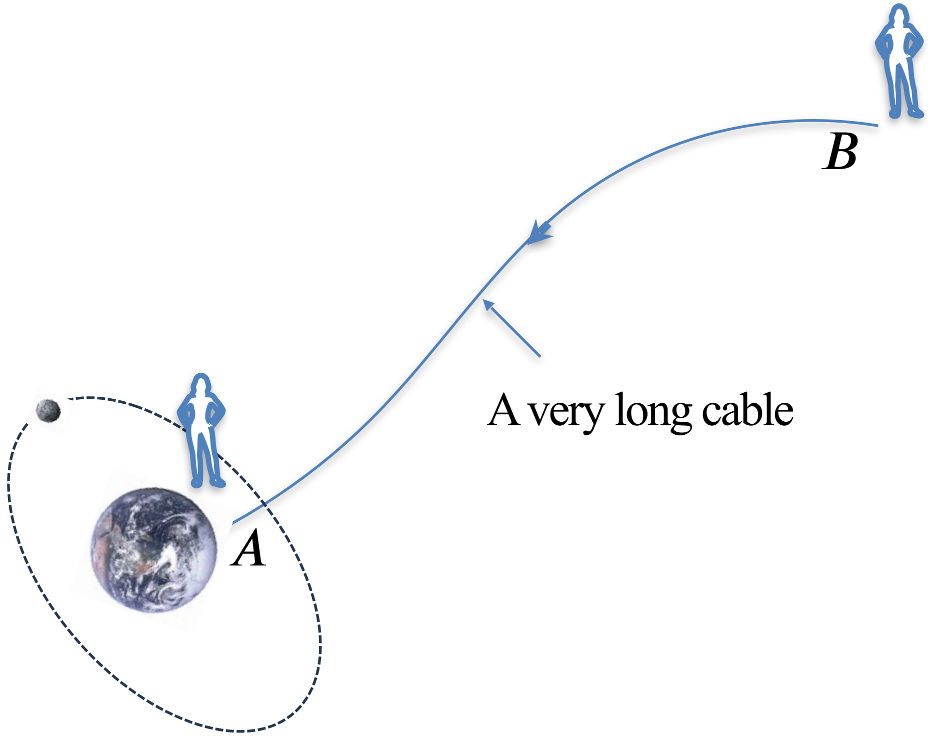
4) Aim to pick a local coordinate frame in which the dynamics of test bodies may presented in a simple form.

5) Metric whose components are the Newtonian potentials† is used to relate proper time and coordinate time.

† Also see IAU resolutions 1991, 2000

Clock comparison in GR: What are we measuring?

Thought experiment: A tale of two identical clocks at different locations (mind the cable!)



Send clock B signal through a cable that is not allowed to move during this experiment.

What is a clock?
A device that locks to a frequency source and tracks it based on a definition. In the simplest form: Count cycles. When a fixed number is reached, increment the counter by a unit and repeat.

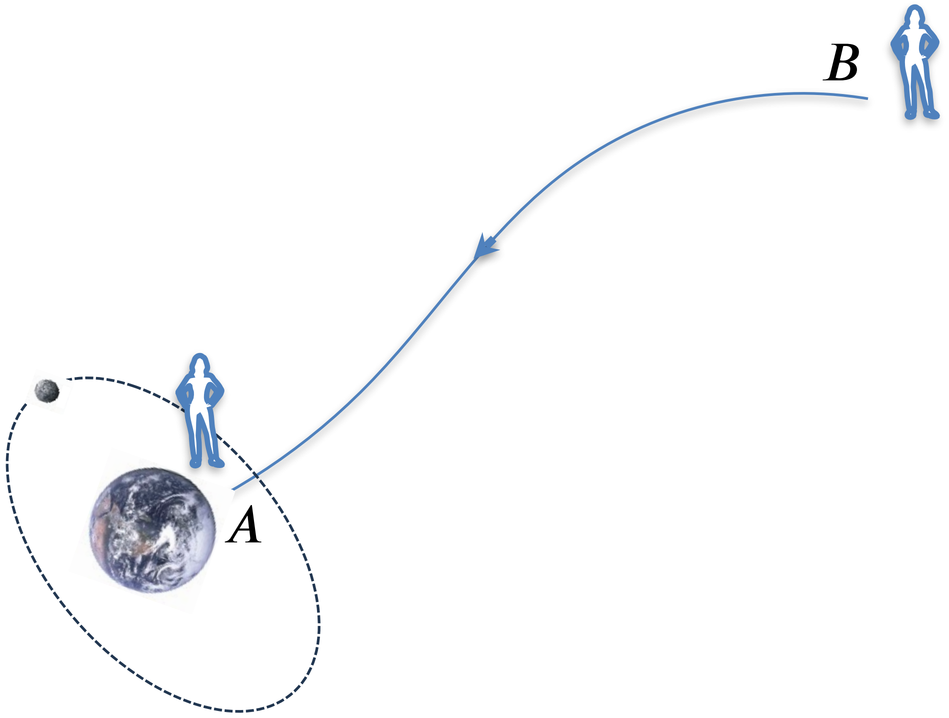
Proper frequency (and time) depend on the gravitational potential at the location of the clock and the motion of the clock in space.

Since there are no long cables connecting every point in spacetime, we have to establish a coordinate time locally that is operationally convenient.

Coordinate frequency is conserved. That is because the separation between successive waveforms or pulses constructed with such signals stays fixed (in spacetime) if you don't move the cable.

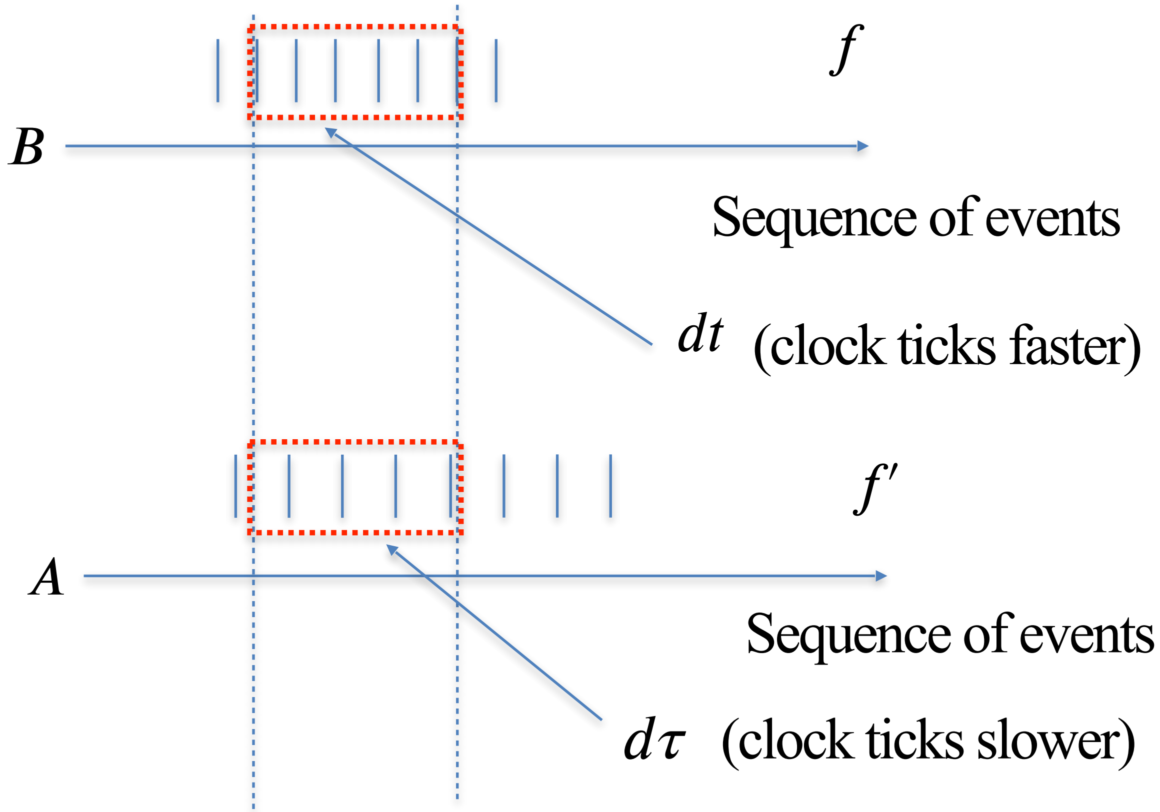
Clock comparison in GR: What are we measuring?

Thought experiment: A tale of two identical clocks at different locations (mind the cable!)



There is a distinction between transporting the clock versus transporting the clock signal—relativistically speaking!

Pulses generated by clocks A and B using their proper frequencies can be compared at the location of clock A.



Clock comparison in GR: What are we measuring?

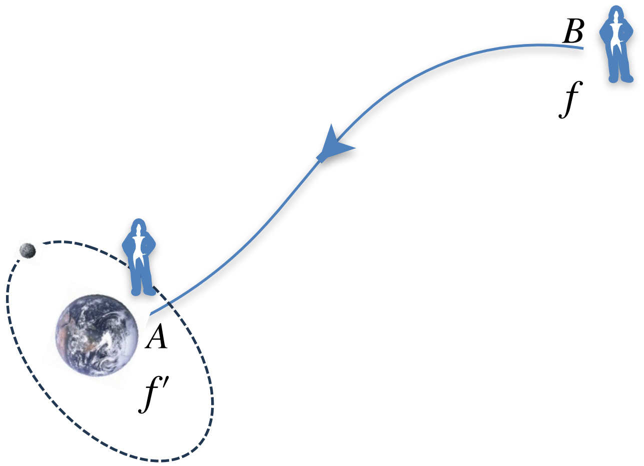
Thought experiment: A tale of two identical clocks at different locations (mind the cable!)

$f' \rightarrow$ measured frequency

$f \rightarrow$ coordinate frequency

$d\tau \rightarrow$ proper time interval

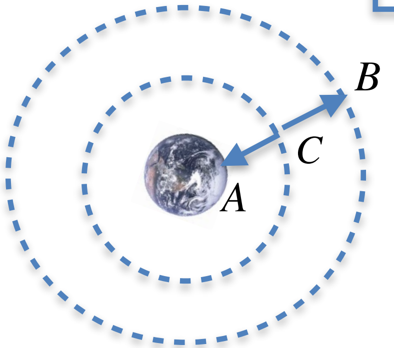
$dt \rightarrow$ realizes coordinate time interval



$$\frac{d\tau}{dt} - 1 = \frac{f' - f}{f}$$

if B is moved to infinity (sufficiently far away away from any gravitational field)

This is the rate offset that is of interest to us for the calculations.



To match the measured frequency of clock C with clock B, apply positive correction to C

To match the measured frequency of clock C with clock A, apply negative correction to C

Clock comparison in SR vs GR (just for appreciating & not understanding!)

SR: two frames with relative velocity:

$$\frac{f_r}{f_0} = \frac{(1 - v^2/c^2)^{1/2}}{(1 - \vec{v} \cdot \vec{r}/c)}$$

GR: three frames with test objects in Keplerian orbits (gravitational potential)

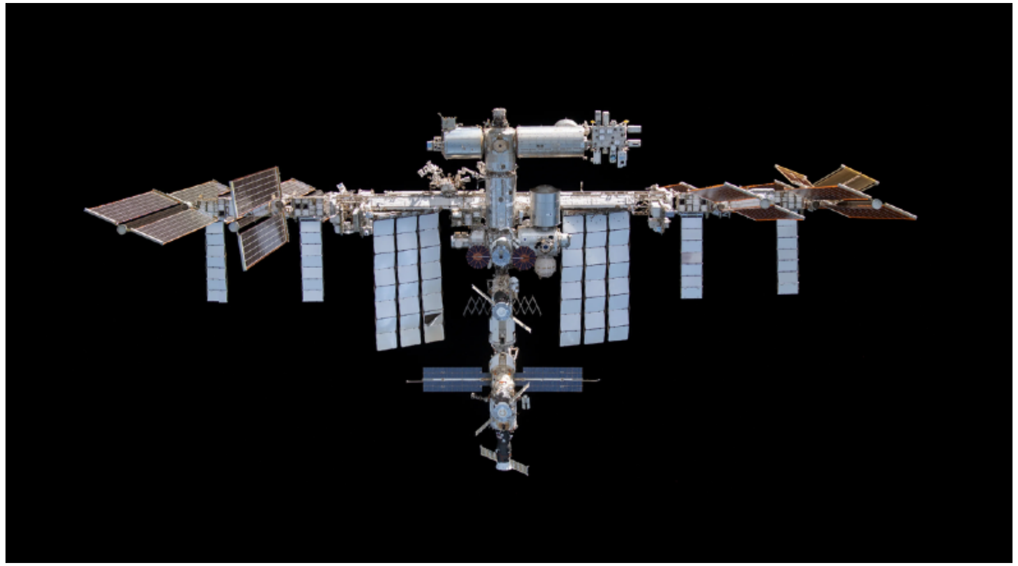
$$\frac{f_B}{f_0} = 1 + \left(\frac{\phi_E - \phi_S}{c^2}\right) - \frac{V_G^2}{2c^2} - \frac{U^2}{2c^2} + \frac{Q_u^2}{c^2} - \frac{n^2(1-e^2)(x_B^2 + z_B^2)}{2c^2(1 - e\cos E)^4(1 + e\cos E)^2} - \frac{\omega_E^2(x_E^2 + y_E^2)}{2c^2}$$

$$U = \frac{n(1-e^2)^{1/2}(x_B k_z^B - z_B k_x^B)_u}{k^B(1 - e\cos E)^2(1 + e\cos E)} \qquad Q_u = \frac{\omega_E(x_E k_y^E - y_E k_x^E)_u}{k^E}$$

[Ashby, Allison & Patla 2025, in preparation]

GPS satellite and ISS comparison

GPS III, Credit: USAF



ISS, Credit: NASA

Height above the earth =20000 km
Rate $=+45.69\ \mu s$ /day

Speed = 14000 km/hr
Rate $= -7.27\ \mu s$ /day

GPS clocks will tick faster (compared to clocks on the geoid) by $\sim 38.4\ \mu s$ /day

Height above the earth = 411.863 km
Rate $=+3.78\ \mu s$ /day

Speed = 27582.68 km/hr
Rate $= -28.21\ \mu s$ /day

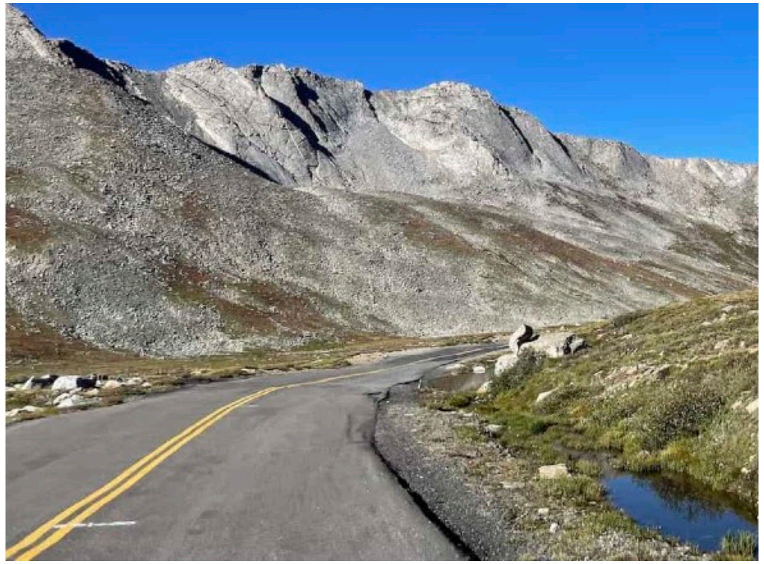
ISS clocks will tick slower (compared to clocks on the geoid) by $\sim 24.4\ \mu s$/day

Clocks on Mount Blue Sky, CO / Elevation 14,265ft ?


NEWS COVID-19 SCPMC CALENDAR SCHEDULE ABO

Colorado Springs man becomes fourth person to push a peanut up Pikes Peak with his nose

By Abigail Beckman · Jul. 15, 2022, 11:59 am



Comparing with Gravity Probe A (GPA) [Vessot , R.F.C. et al. 1980, PRL]

	Clock on Mt Evans, 202?	GPA, 1976
Height difference (km)	2.75	10,000
Experiment duration (hr)	~ 10	~ 1
Oscillator uncertainty, $\delta\left(\frac{\Delta f}{f}\right)$	10^{-18}	10^{-14}
ϵ (theory)	6 parts per million	25 parts per million
ϵ (measured)	~12 parts per million?	~125 parts per million

Time on the moon

THE ASTRONOMICAL JOURNAL, 168:112 (14pp), 2024 September https://doi.org/10.3847/1538-3881/ad643a

CrossMark

A Relativistic Framework to Estimate Clock Rates on the Moon

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Received 2024 April 11; revised 2024 July 15; accepted 2024 July 15; published 2024 August 12

- Using a freely falling coordinate frame centered at the Earth-Moon barycenter.
- Assumes locations of ideal clocks on the surfaces of the Earth and Moon that makes it easier to compare their proper times.
- Calculate proper time difference between clocks on the Moon and the Earth.
- Compute proper time differences between clocks located at the Earth-Moon Lagrange points and on Earth.

From postulation to calculation (just for appreciating & not understanding!)

Taylor series expansion

$$X^\mu = X^\mu(P) + \frac{dX^\mu}{d\lambda} + \frac{1}{2}\frac{d^2X^\mu}{d^2\lambda} + \frac{1}{6}\frac{d^3X^\mu}{d^3\lambda} + \cdots$$

Parallel transport equation

$$\frac{d^2X^\mu}{d^2\lambda} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\lambda}\frac{dX^\beta}{d\lambda} = 0$$

$$-ds^2 = -\left(1 + \frac{2\Phi_e}{c^2} + \frac{2\Phi_m}{c^2} + \frac{2\Phi_s}{c^2}\right)(dX^0)^2 + \left(1 - \frac{2\Phi_e}{c^2} - \frac{2\Phi_m}{c^2} - \frac{2\Phi_s}{c^2}\right)(dX^2 + dY^2 + dZ^2)$$

Solar system barycentric coordinates → dX^μ

Earth-Moon CM coordinates → dx^μ

$$X^0 = \int_0^{x^0}\left(1 - \frac{\Phi_s(cm)}{c^2} + \frac{V(cm)^2}{2c^2}\right)dx^0 + \frac{\boldsymbol{V}_{cm}\cdot\boldsymbol{r}}{c};$$

$$X^k = \int_0^{x^0}\frac{V^k_{cm}}{c}dx^0 + x^k\left(1 + \frac{\Phi_s(cm)}{c^2} - \frac{\boldsymbol{A}_{cm}\cdot\boldsymbol{r}}{c^2}\right) + \frac{r^2A^k_{cm}}{2c^2} + \frac{V(cm)^k\boldsymbol{V}_{cm}\cdot\boldsymbol{r}}{2c^2}.$$

$$g_{\alpha\beta} = \frac{\partial X^\mu}{\partial x^\alpha}\frac{\partial X^\nu}{\partial x^\beta}G_{\mu\nu}$$

$$-ds^2 = -\left(1 + \frac{2\Phi_m}{c^2} + \frac{2\Phi_t}{c^2}\right)(dx^0)^2 + \left(1 - \frac{2\Phi_m}{c^2} - \frac{2\Phi_t}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

Time on the way to the Moon (or anywhere)

Proper time

No. 36

IERS Technical Note

10 General relativistic models for space-time coordinates and equations of motion

$$\frac{d\tau_A}{dTT} = 1 + L_G - 1/c^2 \left[\mathbf{v}_A^2/2 + U_{\mathrm{E}}(\mathbf{x}_A) + V(X_{\mathrm{A}}) - V(X_{\mathrm{E}}) - x_A^i \partial_i V(X_{\mathrm{E}}) \right] \quad (10.8)$$

Coordinate time

Solar system barycenter

Earth center

$V \rightarrow$ Sum of Newtonian potentials of solar system bodies excluding earth

$U_E \rightarrow$ Newtonian potential of earth

$\mathrm{v}_A \rightarrow$ Spacecraft velocity in GCRS

$X_E \rightarrow$ CM coordinates of the earth

$X \rightarrow$ BCRS

$E \rightarrow$ Earth

$x \rightarrow$ GCRS

$A \rightarrow$ Spacecraft

For estimating rates, you only need components of position and velocity vector of the spacecraft.

Time on the way to the Moon (or anywhere)

Measurements/data from: GPS, WAAS, Accelerometers, crosslink, XNAV, terrain maps, ranging. For example, NASA's TDRS in use for more than 40 years.

Models

Geopotential
Solar system bodies
Acceleration noise
Atmosphere
Terrestrial dynamics

Space segment

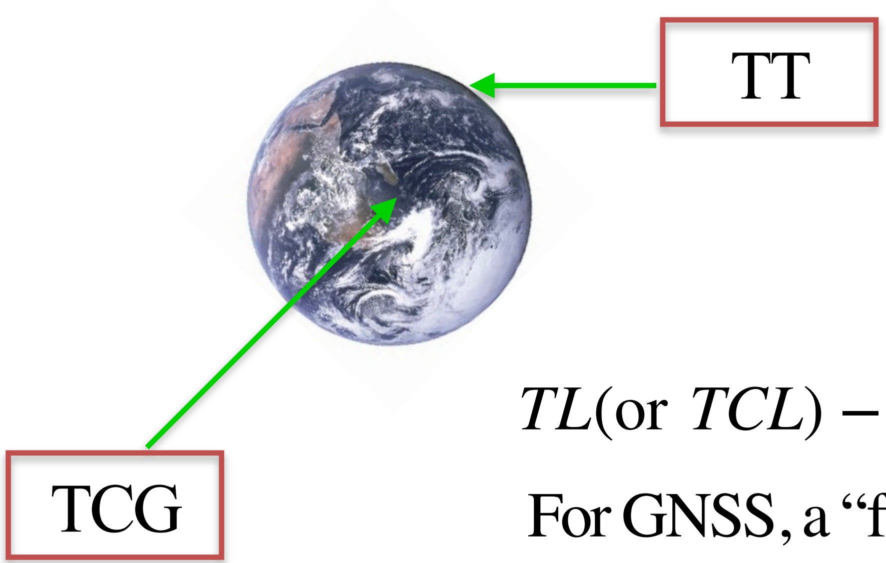
Extended Kalman Filter (EKF)
Noise models

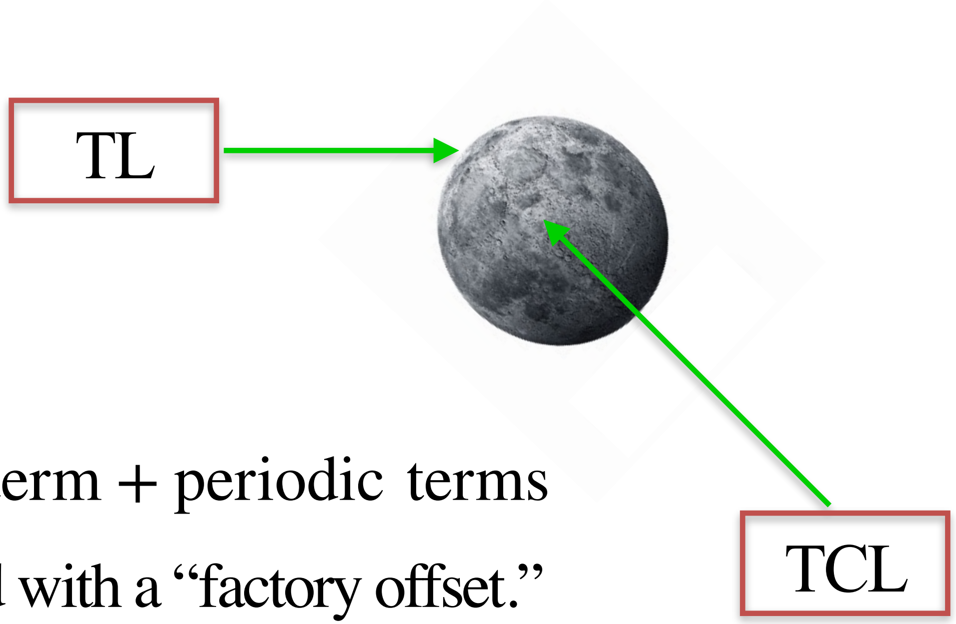
Data processing
Analysis

Ground segment

$$\begin{pmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{pmatrix} \rightarrow \frac{d\tau_A}{dTT} - 1$$

Realizing Lunar Coordinated Time (LTC)





$$TL \leftrightarrow TT$$
$$TCL \leftrightarrow TCG$$

$$TL(\text{or } TCL) - TT = \delta T_L = \text{linear term} + \text{periodic terms}$$

For GNSS, a "fixed" term is corrected with a "factory offset."

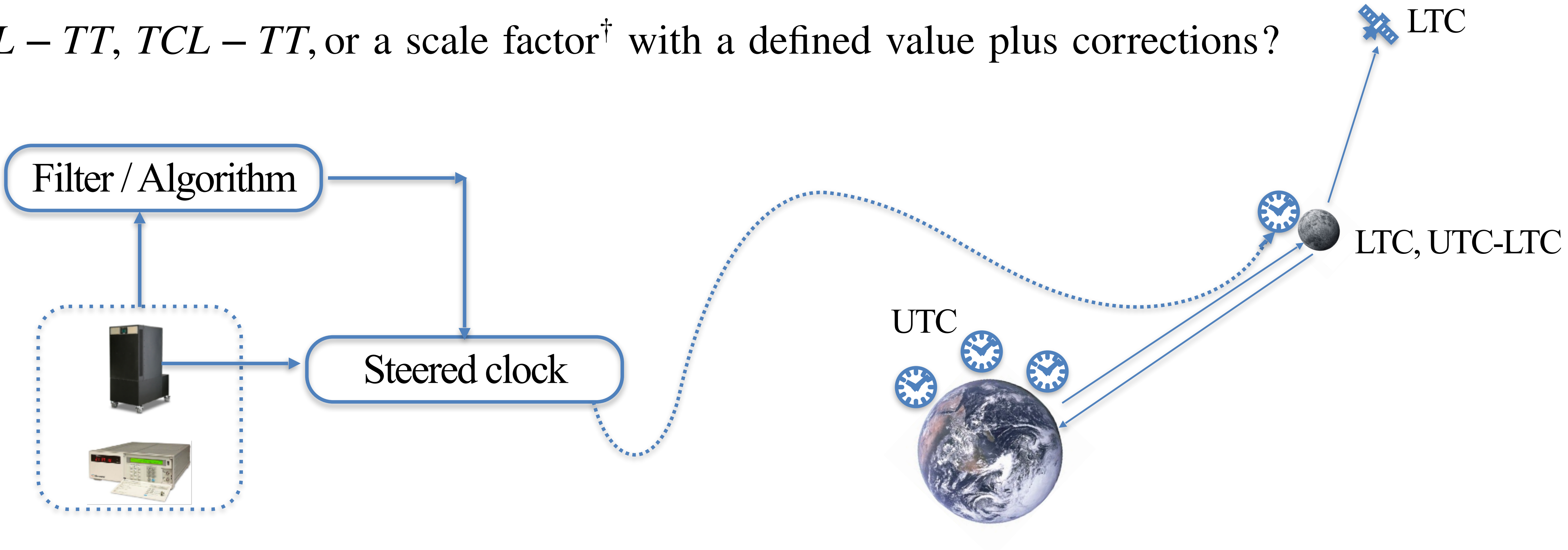
What should be the time standard for cislunar space and for the moon?
Is there any benefit to realizing *TL* on the lunar surface?
Factors to consider include:

a) need for independence from (but traceable to) TT ?
b) how to disseminate δT_L ? What accuracy?
c) what is the optimal way to design to scale up and extend it beyond moon?

Realizing Lunar Coordinated Time (LTC)

How to disseminate lunar time offsets and compare with UTC ?

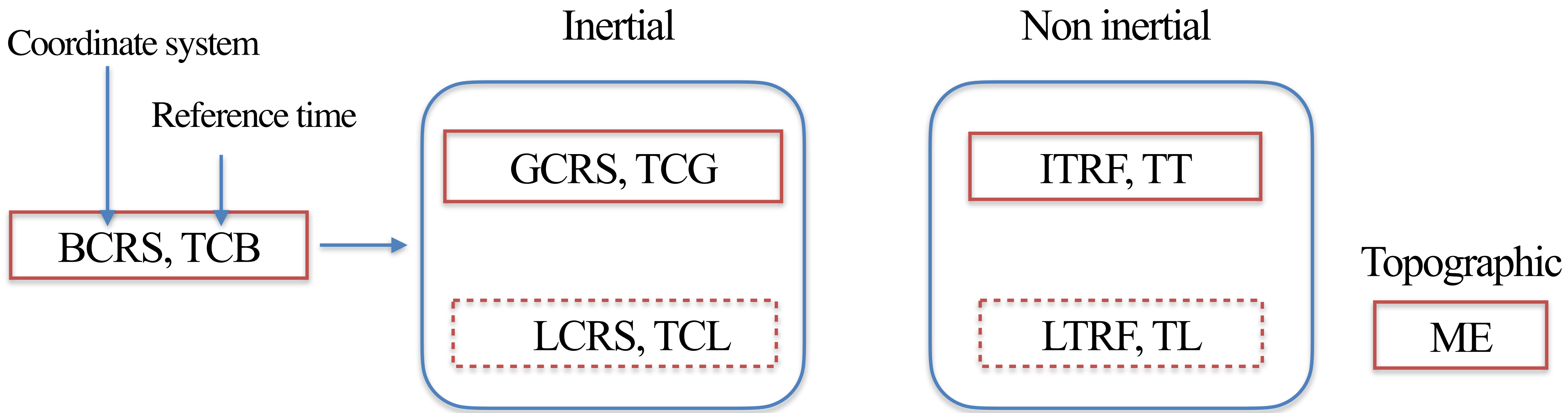
$TL - TT$, $TCL - TT$, or a scale factor† with a defined value plus corrections?



† See Eq. (10) in Ashby & Patla, 2024; Eq. (26) in Kopeikin & Kaplan, 2024

Relativistic effects and reference frames

Mean Earth (†ME) frame for the moon is currently the standard for lunar mapping.



Principal Axis (PA) frame: coordinate axes aligned with principal axes (direction along which moment of inertia is extremal). May be useful depending on the scale and scope of activities planned in circular space.

† X-axis pointing toward the center of the Earth

Relativistic effects and reference frames

Principal Axis (PA) is useful for modeling of the lunar gravity field, precession, nutation.
PA frames are also ideal for optimizing spacecraft dynamics, including attitude control, etc

$$\vec{X}_{\mathrm{PA}} = R(\alpha) \times R(\beta) \times R(\gamma) \times \vec{X}_{\mathrm{ME}}$$

Rotation: Euler angles.

$$\vec{X}_{\mathrm{LCRF}} = P(t) \times N(t) \times R(t) \times \vec{X}_{\mathrm{PA}}$$

Precession, nutation, rotation angle for the Moon.

There is a need for better estimation of the gravitational parameters and gravity field for the moon. The position estimate residuals can a high as a few hundred meters at certain locations on the lunar surface.

Concluding remarks

The time is right for high performance clock comparisons with multiple experiments planned both on the ground and in space.

This April, after 25 years of planning, ACES is scheduled for launch. An ensemble of space qualified H-maser and Cs tube clock on the ISS on a 30-month clock comparison mission.

Understanding relativistic corrections play a vital role in geodesy with clocks, improving gravity field models for the Moon, remote clock comparisons, establishing communication links and deploying relay satellites, design and development of PNT infrastructure, testing fundamental physics —to name a few applications.

In the future, gravitational time delays in signal propagation, higher order multipole moments coupling with tidal potentials, contributions from Coriolis-like effects may have to be considered—depending on specific applications or use cases.

Please stay with us for an in-depth discussion on timing to follow this afternoon.